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Falkner-Skan Solution for Gravity-Driven Film Flow of a Micropolar Fluid

(Penyelesaian Falkner-Skan bagi Aliran Filem Graviti-Terpacu dalam Bendalir Mikrokutub)

KARTINI AHMAD, ROSLINDA NAZAR* & IOAN POP

ABSTRACT

In this paper, the steady Falkner-Skan solution for gravity-driven film flow of a micropolar fluid is theoretically investigated. The resulting nonlinear ordinary differential equations are solved numerically using an implicit finite-difference scheme. The results obtained for the skin friction coefficient as well as the velocity and microrotation or angular velocity profiles are shown in table and figures for different values of the material or micropolar parameter K.

Keywords: Boundary layer; Falkner-Skan solution; gravity-driven film flow; micropolar fluid

ABSTRAK

Dalam makalah ini, penyelesaian Falkner-Skan mantap bagi aliran filem graviti-terpacu dalam bendalir mikrokutub dikaji secara teori. Persamaan pembezaan biasa tak linear yang terhasil diselesaikan secara berangka menggunakan skema beza-terhingga tersirat. Keputusan yang diperoleh bagi pekali geseran kulit beserta profil halaju dan profil mikroputaran atau halaju sudut dipersembahkan dalam jadual dan rajah untuk beberapa nilai parameter bahan atau mikrokutub K.

Kata kunci: Aliran filem graviti-terpacu; bendalir mikrokutub; lapisan sempadan; penyelesaian Falkner-Skan

INTRODUCTION

The concept of classical hydrodynamics is inadequate to describe the existence of microscopic elements in a fluid due to the local microstructure and intrinsic motion of the fluid. Thus the concept of micropolar fluid which was first proposed by Eringen (1966) has taken into account the microstructure of the fluid and the physical characteristics of the fluid itself, i.e. oriented particles suspended in a viscous medium undergoing both translational and rotational motion. The theory of micropolar fluid has received substantial attention during the last few decades due to the importance of such fluid in analyzing the behaviour of animal blood flow, exotic lubricants, flow in capillaries and micro-channels, liquid crystals and colloidal fluids. On top of solving the usual transport equations for the conservation of mass and momentum equations, the micropolar fluid theory requires one to solve an additional transport equation representing the principle of conservation of local angular momentum. Extensive reviews of the theory and applications of micropolar fluids can be found in the review articles by Ariman et al. (1973, 1974) and the books by Łukaszewicz (1999) and Eringen (2001).

Fluid flows which are called film flows occur in many technical processes as well as in nature. Rain wetted roads and erosion are examples of films encountered in environment. In industry, we find films in heat exchangers, evaporators, condensers, and absorption and coating techniques (Kistler & Schweizer 1997; Webb 1994). In most systems, the film does not flow over perfectly flat surface. Therefore, the gravity- driven film flow is one of the most studied systems in hydrodynamics; examples are as in the papers by Andersson and Irgens (1988) and Andersson and Shang (1998) and the references therein. A year later, Andersson and Dahl (1999) studied the gravity-driven flow of a viscoelastic liquid film along a vertical wall. In addition, rigorous mathematical analysis of a boundary layer problem for a third-order nonlinear ordinary differential equation which arose in gravity-driven laminar film flow of power law fluids along vertical walls was done by Zhang et al. (2004). Further, Saouli et al. (2006) investigated the entropy generation in laminar, gravity-driven conducting liquid film with fully developed velocity flowing along an inclined heated plate in the presence of transverse magnetic field, and Andersson et al. (2006) investigated the gravity-driven film flow with variable physical properties. On the other hand, the study of gravity-driven film flow of a liquid film down an inclined wall with three-dimensional doubly periodic corrugations was carried out by Luo and Pozrikidis (2007). The experimental study of gravity-driven film flow of non-Newtonian fluids namely carboxymethyl cellulose (CMC) with three different solution concentrations was done by Haeri and Hashemabadi (2009), while Lan et al. (2010) examined the problem involving three-dimensional steady-developing-laminar-isothermal and gravity-driven liquid film flow adjacent to an inclined plane. The problem was numerically simulated and experimental verification was conducted. Very recently, Beg et al. (2011) examined numerically the problem of steady, gravity-driven, incompressible, hydromagnetic, laminar flow of a viscous, ellectrically conducting, micropolar liquid along an inclined plane subjected to a uniform transverse magnetic field. The computations indicate that increasing micropolarity will elevate the micro-rotation magnitudes but reduces the linear velocity.

The work done by Falkner and Skan (1931) was a starting point of exact solutions for boundary layer flow in a viscous fluid. The paper considered two-dimensional wedge flows by developing a similarity transformation method in which the boundary layer partial differential equation was reduced to a nonlinear third-order ordinary differential equation, which is well-known as the Falkner–Skan equation. Later, Hartree (1937) studied this equation numerically. There is abundance of literature on the solution of the Falkner-Skan equation, such as the papers by Rajagopal et al. (1983), Brodie and Banks (1986), Asaithambi (1997), Zaturska and Banks (2001), Kuo (2003), Pantokratoras (2006) and Ishak et al. (2007), among others. Some recent literatures can be found in Elgazery (2008) where he solved the Falkner–Skan boundary layer equation with the effect of magnetic field in a porous medium, while Magyari (2009) considered the case of Falkner-Skan flows past stretching boundaries, and Abbasbandy and Hayat (2009) studied the two-dimensional steady boundary layer flow of an electrically conducting viscous fluid in the presence of a magnetic field, to name just a few. However, to the best of our knowledge, the Falkner-Skan solution for gravity-driven film flow was first and only investigated by Andersson and Ytrehus (1985). They found that the Falkner-Skan equation can be used to solve problem in developing film flow on a vertical wall especially in aerodynamic boundary layer for the case of $m = 1/2$ which corresponds to flow along a wedge with an opening angle $2\pi/3$. Motivated by the work done in Andersson and Ytrehus (1985), the present paper will consider the Falkner-Skan solution for gravity–driven film flow of a micropolar fluid.

ANALYSIS

Consider the steady flow in a gravity-driven thin liquid film of a micropolar fluid falling downwards along a smooth vertical surface, as shown schematically in Figure 1, where x and y are the Cartesian coordinates measured along the surface and normal to it, respectively. It is assumed that the buoyancy and the surface tension effects are neglected. Under these assumptions, the system of boundary layer equations governing the problem under consideration can be written as (Andersson & Ytrehus 1985; Rees & Bassom 1996)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{(\mu+\kappa)}{\rho}\frac{\partial^2 u}{\partial y^2}+\frac{\kappa}{\rho}\frac{\partial N}{\partial y}. \tag{2}$$

$$\rho j\left(u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}\right)=\frac{\partial}{\partial y}\left(\gamma\frac{\partial N}{\partial y}\right)-\kappa\left(2N+\frac{\partial u}{\partial y}\right). \tag{3}$$

$$u\frac{\partial j}{\partial x}+v\frac{\partial j}{\partial y}=0. \tag{4}$$

and subject to the boundary conditions:

$$\begin{aligned}&u=v=0.\quad N=-\frac{1}{2}\frac{\partial u}{\partial y},\ j=0\quad \text{at}\quad y=0,\\ &u=U(x),\ N=0,\ \text{as}\quad y\to\infty,\end{aligned} \tag{5}$$

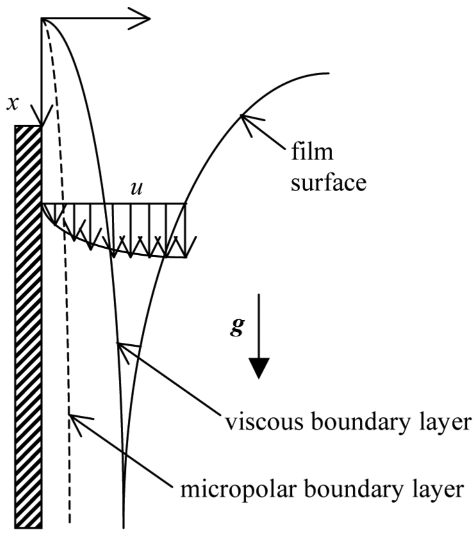

FIGURE 1. Physical model and coordinate system

where u and v are the velocity components along the x and y-axes, respectively, N is the component of the microrotation vector normal to the $x - y$ plane, j is the microinertia density, γ is the spin gradient viscosity, κ is the vortex viscosity, μ is the dynamic viscosity, ρ is the fluid density, g is the gravitational acceleration and $U(x)$ is the free stream velocity. Further we assume that $\gamma(x, y)$ is given as (Yacob & Ishak 2010):

$$\gamma=\left(\mu+\frac{\kappa}{2}\right)j(x,y)=\mu\left(1+\frac{K}{2}\right)j(x,y), \tag{6}$$

where $K = \kappa/\mu$ is the micropolar or material parameter so that the micropolar fluid field can predict the correct behaviour in the limiting case when the microstructure effects become negligible and the total spin N reduces to the angular flow velocity or flow vorticity**.** It should be mentioned that relation (6) has been established by Ahmadi (1976) and Kline (1977), and it has been used by many researchers working on micropolar fluids. It is also worth mentioning that the case $K = 0$ describes the classical Navier-Stokes equations for a viscous and incompressible fluid.

The quasi-one-dimensional flow between the momentum boundary layer and the wavefree interface bordering the constant-pressure atmosphere is assumed to be irrotational (inviscid flow) with downward velocity $U(x)$. The one-dimensional equations of motion for the inviscid flow then becomes (Andersson & Ytrehus 1985)

$$U\frac{dU}{dx} = g, \tag{7}$$

which is equivalent to the one-dimensional version of (2). Assuming zero velocity (and infinite film thickness) at the entrance $x = 0$, the simple free stream solution:

$$U(x) = (2gx)^{1/2}, \tag{8}$$

is readily obtained by the integration of (7). Incidentally, the inviscid solution (8) belongs to the important class of free streams $U(x) \sim x''$, discovered by Falkner-Skan (1931), which arises in aerodynamic boundary layer flow along wedge-shaped bodies. Having this in view, we look for a similarity solution of equation (1) to (4) of the following form:

$$\begin{gathered}\psi = (4U\nu x/3)^{1/2} f(\eta),\ \ N = U(3U/4\nu x)^{1/2}\, h(\eta)\\ j = \frac{\nu x}{U} i(\eta),\ \ \eta = (3U/4\nu x)^{1/2}\, y,\end{gathered} \tag{9}$$

where ν is the kinematic viscosity and $\psi(x, y)$ is the stream function, which is defined in the usual way as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, and automatically satisfy equation (1). Substituting (9) into equation (2) to (4), the following system of ordinary differential equations is obtained:

$$(1+K)f''' + f f'' + \frac{2}{3}\left(1 - f'^2\right) + K h' = 0. \tag{10}$$

$$\left(1+\frac{K}{2}\right)(ih')' + i\left(f h' - \frac{1}{3} f' h\right) - \frac{4}{3}K\left(2h + f''\right) = 0. \tag{11}$$

$$2f'i - 3fi' = 0. \tag{12}$$

subject to the boundary conditions:

$$\begin{gathered}f(0) = f'(0) = 0,\ \ h(0) = -\frac{1}{2}f''(0),\ \ i(0) = 0\\ f'(\infty) = 1,\ \ h(\infty) = 0,\end{gathered} \tag{13}$$

where primes denote differentiation with respect to η. It is worth mentioning that setting $K = 0$ (Newtonian fluid) in (10) and (11) recovers the problem considered by Andersson and Ytrehus (1985). Further, integrating equation (12) with the boundary conditions (13), we get:

$$i = S f^{2/3}, \tag{14}$$

where S is a constant of integration.

The physical quantity of interest is the skin friction coefficient C_f, which is defined as:

$$C_f = \frac{\tau_w}{\rho U^2/2}, \tag{15}$$

where the wall shear stress τ_w is defined as:

$$\tau_w = \left[(\mu+\kappa)\frac{\partial u}{\partial y} + \kappa N\right]_{y=0}. \tag{16}$$

Using the variables (9), we get:

$$C_f \,\mathrm{Re}_x^{1/2} = \sqrt{3}\left(1+\frac{K}{2}\right)f''(0), \tag{17}$$

where is the local Reynolds number which is defined as $\mathrm{Re}_x = Ux/\nu$.

Results and Discussion

The ordinary differential equation (10) to (12) subject to the boundary conditions (13) are solved numerically using an implicit finite-difference scheme known as the Keller–box method as discussed in the book by Cebeci & Bradshaw (1988). The solution procedure can be summarized by the following four steps:

1. Reduce (10) to (12) to a first-order system.
2. Write the difference equations using central differences.
3. Linearize the resulting algebraic equations by Newton's method and write them in matrix-vector form.
4. Using block-tridiagonal-elimination technique, solve the linear system obtained.

The values of the step size $\Delta\eta$ in η and the edge of boundary layer (η_∞) have to be adjusted for different values of parameters to maintain accuracy. Throughout this study, we considered the value of $\Delta\eta = 0.02$ and was found to be satisfactory for a convergence criterion of 10^{-5} which gives four decimal places accuracy. On the other hand, the edge of the boundary layer chosen was between 5 and 10. It is worth mentioning that the numerical scheme used in the present study has been proven to be unconditionally stable and it is also the most flexible of the common methods, being easily adaptable to solving equations of any order (Cebeci & Bradshaw 1988). To verify the accuracy of the present method, the value of the reduced skin friction coefficient $f''(0)$ is compared with that reported by Andersson and Ytrehus (1985) for $K = 0$ (Newtonian fluid) as presented in Table 1, and it is found to be in good agreement. Table 1 contains also the numerical values of the skin friction coefficient $C_f\,\mathrm{Re}_x^{1/2}$ for various values of the material parameter K, namely $K = 0$ (Newtonian fluid), 1, 2 and 3 (micropolar fluid) with the constant of integration $S = 1$. It is seen from Table 1 that the reduced skin friction coefficient $f''(0)$ is lower for micropolar fluid than Newtonian fluid, while the trend

TABLE 1. Values of $f''(0)$ and $C_f \mathrm{Re}_x^{1/2}$ for various values of K with S=1

K	Andersson and Ytrehus (1985)	$f''(0)$	$C_f \mathrm{Re}_x^{1/2}$
0	1.03890	1.0397	1.8008
1		0.8485	2.2045
2		0.7348	2.5454
3		0.6572	2.8458

for the skin friction coefficient $C_f \mathrm{Re}_x^{1/2}$ is vice versa. This is due to the coefficient $\sqrt{3}\left(1+K/2\right)$ in the skin friction coefficient $C_f \mathrm{Re}_x^{1/2}$. It is also observed that as K increases, $f''(0)$ decreases while $C_f \mathrm{Re}_x^{1/2}$ increases.

Figure 2 and 3 show the velocity profiles $f'(\eta)$ and the microrotation or angular velocity profiles $h(\eta)$ for various values of K, respectively, while Figure 4 and 5 illustrate the effects of S towards the velocity and angular velocity profiles for various values of K, namely $K = 0.01$, 1 and 3. It is seen from Figure 2 that the velocity profiles $f'(\eta)$ always start at $\eta = 0$ with $f'(0) = 0$, and once it reaches a certain thickness of the boundary layer, the velocity at this point onwards will maintain the value of 1 asymptotically, as given by the boundary conditions (13), i.e. $f'(\infty) = 1$. These results show that increasing the material parameter K, namely transition from the Newtonian fluid to micropolar fluid leads to the deceleration of the velocity, which in turn decreases the velocity gradient at the surface $\eta = 0$, and hence produces decrement in the reduced skin friction coefficient $f''(0)$. In addition, the boundary layer thickness increases as the material parameter K increases. It is seen from Figure 2 that the velocity gradients at the surface $\eta = 0$ are all positives, which physically corresponds to the fluid exerts a drag force on the surface. Further, it is seen from Figure 3 that the angular velocity $h(\eta)$ validates the boundary conditions (13) at the surface and at the edge of boundary layer, and $h(0)$ is slightly low for small values of K in the beginning of the layer until it reaches a certain value of η and the boundary layer thickness increases as the material parameter K increases. The microrotation or angular velocity exhibits different characteristic than the velocity and becomes zero far away from the surface which satisfies the boundary condition $h(\infty) = 0$. This means that at the edge of boundary layer, the microrotation or angular velocity is zero.

On the other hand, Figures 4 and 5 show the effects of the constant of integration S on the velocity and microrotation profiles, respectively, as K is fixed to several values, i.e. 0.01, 1 and 3 (micropolar fluids). It should be pointed out that the parameter S does not have any significant impact towards the behaviour of the velocity and microrotation profiles, regardless of the values of the micropolar or material parameter K. It was observed that variation in S, i.e. different values of S, gave almost exactly the same profiles for each value of K.

CONCLUSION

In the present paper, we have studied theoretically the Falkner-Skan solution for gravity-driven film flow of a micropolar fluid. The governing partial differential equations were transformed using suitable transformation into ordinary differential equations and hence were solved numerically using the finite–difference scheme known as the Keller–box method. Numerical results for the reduced skin friction coefficient $f''(0)$, the skin friction coefficient $C_f \mathrm{Re}_x^{1/2}$, the velocity profiles $f'(\eta)$ and the microrotation or angular velocity profiles $h(\eta)$ were presented in a table and some graphs. A comparison with the result reported by Andersson and Ytrehus (1985) was made when the material parameter $K = 0$ (Newtonian fluid) and the agreement was

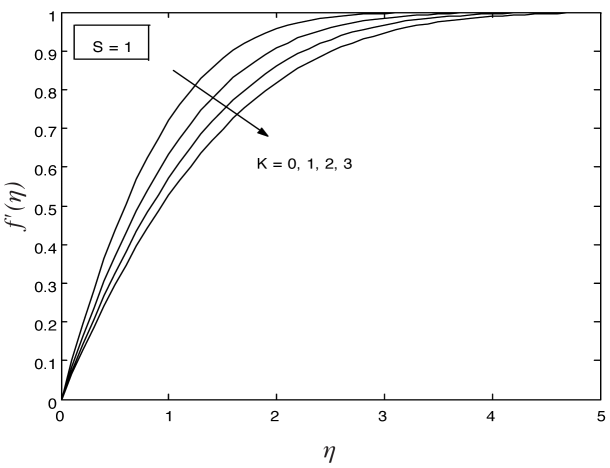

FIGURE 2. Velocity profiles $f'(\eta)$ for various values of K when S=1

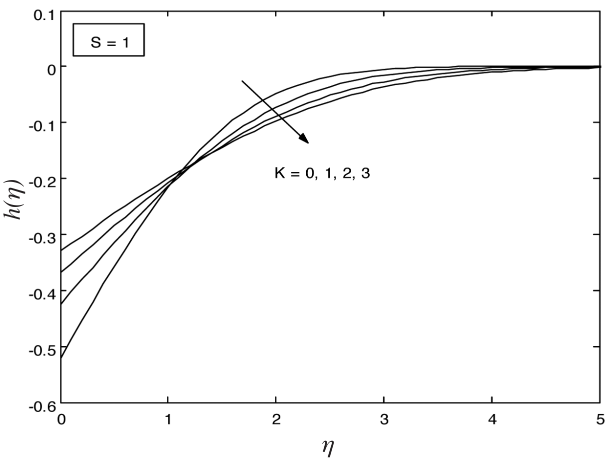

FIGURE 3. Microrotation or angular velocity profiles $h(\eta)$ for various values of K when S=1

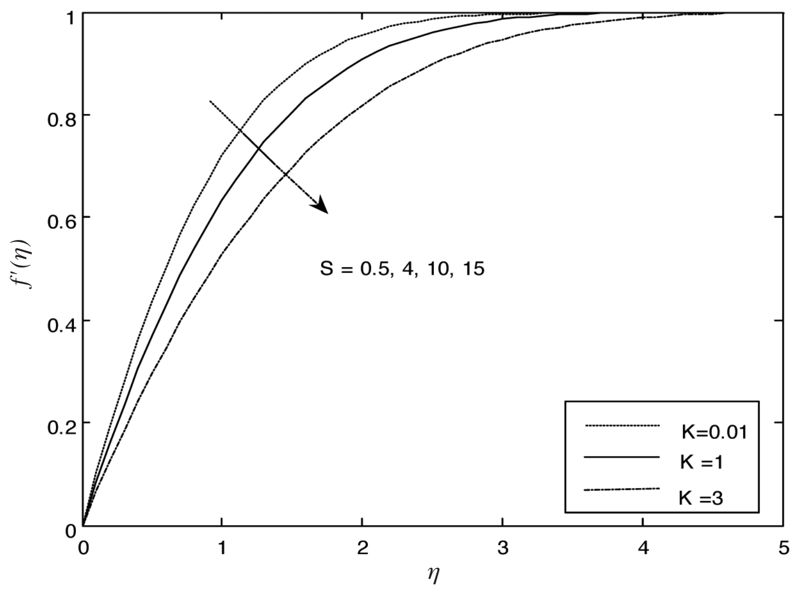

FIGURE 4. Velocity profiles $f'(\eta)$ for various values of S and K

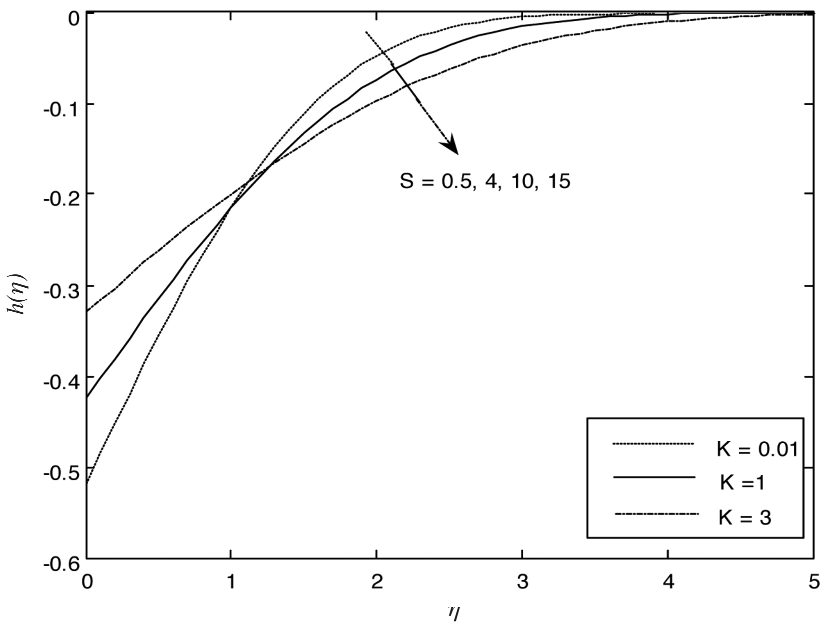

FIGURE 5. Microrotation or angular velocity profiles $h(\eta)$ for various values of S and K

very good.

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Kartini Ahmad
Centre for Foundation Studies
International Islamic University Malaysia
Jalan Universiti, 46350 Petaling Jaya
Selangor, Malaysia

Roslinda Nazar*
Centre for Modelling & Data Analysis
School of Mathematical Sciences
Faculty of Science & Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor, Malaysia

Roslinda Nazar*
Solar Energy Research Institute
Universiti Kebangsaan Malaysia
43600 Bangi
Selangor, Malaysia

Ioan Pop
Faculty of Mathematics
University of Cluj
R-3400 Cluj, CP 253
Romania

*Corresponding author; email: rmn@ukm.my

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